A remark on a new category of supermanifolds

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Abstract. *In a previous paper a new category of supermanifolds, called $\mathcal{G}$-supermanifolds, was introduced. The objects of that category are pairs $(M, \mathcal{A})$, with M a topological space and $\mathcal{A}$ a suitably defined sheaf of $\mathbb{Z}_2$-graded commutative B_L-algebras, B_L being a Grassmann algebra with L generators. In this note we complete the analysis of that category by showing that $\mathcal{A}$ is isomorphic with the sheaf of $\mathcal{G}$-maps $M \rightarrow B_L$.*

1. INTRODUCTION

In a previous paper [1] a new category of supermanifolds, called $\mathcal{G}$-supermanifolds, was introduced, according to the following motivations. In the original «geometric» approach to supermanifolds [2-3], as opposed to the «algebraic» one which yields the Berezin-Kostant category of graded manifolds [4-5], one considers topological spaces M which are locally modelled on the space

$$B_L^{m,n} = (B_L)_0^m \times (B_L)_1^n$$

where $B_L = (B_L)_0 \oplus (B_L)_1$ is a Grassmann algebra with L generators, endowed with the vector space topology. The pair of non-negative integers (m, n) is called

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the dimension of M. The atlas used to model M has transition functions fulfilling a suitable «supersmoothness» condition, i.e. they are G^∞ functions [3].

However, Rothstein [6] showed that the sheaf of graded (1) derivations of the structure sheaf of a G^∞ supermanifold M (i.e. the graded tangent sheaf of M) is not locally free, and proposed an alternative axiomatic definition of supermanifold, which is again given in algebraic terms and ensures that a supermanifold has all nice properties one expects.

On the other hand, in [7] Rogers tried to refine the notion of supersmooth functions, so as to be able to stick to the definition of supermanifolds in terms of transition functions. While it is true that the objects of the resulting category, called GH^∞ supermanifolds, have a locally free graded tangent sheaf, in [1] it was proved that the tangent spaces at the various points of the supermanifold are not isomorphic to each other, so that there is no sensible notion of graded tangent bundle to a GH^∞ supermanifold. More generally, there is no good theory of super vector bundles with standard fibre over a GH^∞ supermanifold [1]. In accordance with [6], it comes out that a GH^∞ supermanifold is not a supermanifold in Rothstein's sense.

A GH^∞ supermanifold can be turned into an object of a larger category, that of $\mathcal{G}$-supermanifolds, which is a special case of Rothstein's supermanifolds [1]; the definition of this new category is reviewed in Section 2. The aim of this paper is to fill a gap in the theory as expounded in [1]; we shall show that, if $(M, \mathcal{A})$ is a $\mathcal{G}$-supermanifold, the sheaf $\mathcal{A}$ is isomorphic with the sheaf of germs of $\mathcal{G}$-maps $M \to B_L$. This point is a crucial one in the development of a theory of super vector bundles [1, 8, 9, 10].

2. PRELIMINARIES

In this Section we summarize the basic notations and definitions concerning GH^∞ and $\mathcal{G}$-supermanifolds. Motivations and further details are to be found in [1].

Let Ξ_L denote the set of strictly increasing sequences of natural numbers between 1 and L, i.e.

$$\Xi_L = \cup_{r=1}^{L} \{\mu : \{1, \dots, r\} \to \{1, \dots, L\} \text{ strictly increasing}\}.$$

If $\{e_i : 1 \le i \le L\}$ is a basis for $\mathbb{R}^L$, then

$$\{\beta_\mu \equiv e_{\mu(1)} \wedge \dots \wedge e_{\mu(r)} : \mu \in \Xi_L\}$$

is a basis for B_L, here identified with $\wedge[\mathbb{R}^L]$. Let N_L be the ideal of nilpotents of B_L; then $B_L = \mathbb{R} \oplus N_L$, and the projections $\sigma : B_L \to \mathbb{R}$, $s : B_L \to N_L$ are called

(1) In the following by «graded» we always mean «$\mathbb{Z}_2$-graded.»

body and *soul* map respectively. The cartesian product B_L^{m+n} has a natural structure of graded B_L-module obtained by letting

$$B_L^{m+n} = \left[(B_L)_0^m \times (B_L)_1^n\right] \oplus \left[(B_L)_1^m \times (B_L)_0^n\right] \equiv B_L^{m,n} \oplus B_L^{\bar{m},\bar{n}} \tag{2.1}$$

A body map $\sigma^{m,n} : B_L^{m,n} \to \mathbf{R}^m$ is defined by letting $\sigma^{m,n}(x^1 \dots x^m, y^1 \dots y^n) = (\sigma(x^1) \dots \sigma(x^m))$. $B_L^{m,n}$ will be considered as a topological space with its vector space topology.

For any smooth manifold X, denote by $C_L^\infty(W)$ the sections over $W \subset X$ of the sheaf of B_L-valued C^∞ functions on X. Given two positive integers L and L', with $L' \leq L$, a morphism of graded algebras

$$Z_{L',L} : C_{L'}^\infty(U) \to C_L^\infty((\sigma^{m,0})^{-1}(U))$$

is defined for any $U \subset \mathbf{R}^m$ by letting [7]

$$Z_{L',L}(f)(x^1 \dots x^m) =$$

$$\sum_{i_1 \dots i_m = 0}^{L} \frac{1}{i_1! \dots i_m!} (\partial_1^{i_1} \dots \partial_m^{i_m} f)|_{(\sigma(x^1) \dots \sigma(x^m))} s(x^1)^{i_1} \dots s(x^m)^{i_m}.$$

$Z_{L',L}$ is injective for all U; its image is decreed to be the space of GH^∞ functions of even variables on $(\sigma^{m,0})^{-1}(U)$.

On $(\sigma^{m,n})^{-1}(U)$, where U is open in $\mathbf{R}^m$, the algebra $\mathcal{GH}\left((\sigma^{m,n})^{-1}(U)\right)$ is defined as the space of functions having the form

$$F(x^1 \dots x^m, y^1 \dots y^n) = \sum_{\mu \in \Xi_n} F_\mu(x^1 \dots x^m)\, y^\mu \tag{2.2}$$

where $y^\mu \equiv y^{\mu(1)} \dots y^{\mu(r)}$ and $F_\mu \in Z_{L',L}\left(C_{L'}^\infty(U)\right)$. $\mathcal{GH}\left((\sigma^{m,n})^{-1}(U)\right)$ is naturally a graded commutative $B_{L'}$-algebra, so that a sheaf $\mathcal{GH}$ of graded commutative $B_{L'}$-algebras over $B_L^{m,n}$ is defined by letting, for all open sets $V \subset B_L^{m,n}$,

$$\mathcal{GH}(V) = \mathcal{GH}\left((\sigma^{m,n})^{-1}\sigma^{m,n}(V)\right) \tag{2.3}$$

If the condition $L - L' \geq n$ is verified, which we shall henceforth assume when dealing with GH^∞ functions, the derivatives of a GH^∞ function F are uniquely determined by the expansion

$$F(z+h) = F(z) + \sum_{A=1}^{m+n} h^A \frac{\partial F}{\partial z^A}(z) + \sum_{A,B=1}^{m+n} h^A h^B g_{AB}(z,h) \tag{2.4}$$

where $z, h \in B_L^{m,n}$. This implies that the sheaf of graded derivations of $\mathcal{GH}$ is locally free [1,7].

On the contrary, if $L - L' < n$ not all the coefficients of the expansion (2.4) are defined, so that the sheaf of derivations is not locally free [6]. If in particular $L = L'$, the functions are said to be G^∞, and the corresponding sheaf on $B_L^{m,n}$ will be denoted by $\mathcal{G}^\infty$.

DEFINITION 1. *An* $(m;n)$ *dimensional* GH^∞ *supermanifold is a ringed space* $(M,\mathcal{A})$, *with* M *Hausdorff paracompact, locally isomorphic with* $(B_L^{m,n},\mathcal{GH})$.

Obviously, this definition is equivalent to the one in terms of GH^∞ transition functions. In particular, a GH^∞ map is one which is GH^∞ when expressed in terms of local charts.

The sheaf of graded B_L-algebras $\mathcal{G}$ on $B_L^{m,n}$ is defined as

$$\mathcal{G} = \mathcal{GH} \otimes_{B_{L'}} B_L;$$

an «evaluation» map $\delta : \mathcal{G} \to \mathcal{C}_L$, where $\mathcal{C}_L$ is the sheaf of continuous B_L-valued functions on $B_L^{m,n}$, is defined by letting

$$\delta(f \otimes a) = fa \tag{2.5}$$

and extending by linearity. It is easily checked that the image of δ coincides with the sheaf $\mathcal{G}^\infty$ of G^∞ functions on $B_L^{m,n}$

The (formal) partial derivatives of sections of $\mathcal{G}$ are defined according to

$$\frac{\partial}{\partial x^i}(f \otimes a) = \frac{\partial f}{\partial x^i} \otimes a,\ i = 1 \dots m;$$
$$\frac{\partial}{\partial y^\alpha}(f \otimes a) = \frac{\partial f}{\partial y^\alpha} \otimes a,\ \alpha = 1 \dots n.$$

DEFINITION 2. *An* (m,n) *dimensional* $\mathcal{G}$*-supermanifold is a ringed space* $(M,\mathcal{A})$ *such that*

(i) M *is Hausdorff paracompact;*

(ii) $(M,\mathcal{A})$ *is locally isomorphic with* $(B_L^{m,n},\mathcal{G})$;

(iii) there is a morphism $\mathcal{A} \to \mathcal{C}_L^M$ *(where* $\mathcal{C}_L^M$ *is the sheaf of germs of continuous* B_L*-valued functions on* M *) which is compatible with the morphism (2.5) and with condition (ii).*

The requirements (ii) and (iii) mean firstly that any $x \in M$ has a neighbourhood U with a homeomorphism $\psi : U \to W \subset B_L^{m,n}$ such that there exists an isomorphism

$$\mathcal{A}(U) \xrightarrow{\sim} \mathcal{G}(W) \tag{2.6}$$

compatible with restrictions, and secondly that the diagram

$$\begin{array}{ccc} \mathcal{A}(U) & \rightarrow & \mathcal{C}_L^M(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(W) & \rightarrow & \mathcal{C}_L(W) \end{array}$$

commutes. The morphism $\mathcal{A} \to C_L^M$ will be again denoted by δ and its image by $\mathcal{A}^\infty$. It is straightforward to verify that $(M, \mathcal{A}^\infty)$ is a G^∞ supermanifold in the sense of [3].

If $(M, \mathcal{A})$ is a GH^∞ supermanifold, by defining $\tilde{\mathcal{A}} = \mathcal{A} \otimes_{B_{L'}} B_L$ one obtains a $\mathcal{G}$-supermanifold $(M, \tilde{\mathcal{A}})$ which in a sense is the «trivial extension» of $(M, \mathcal{A})$.

3. MAPS OF $\mathcal{G}$-SUPERMANIFOLDS

We wish now to show that, given a $\mathcal{G}$-supermanifold $(M, \mathcal{A})$, the sheaf of germs of $\mathcal{G}$-maps (2) from $(M, \mathcal{A})$ to the $\mathcal{G}$-supermanifold $(B_L, \mathcal{G})$ (where B_L is regarded as $B_L^{1,1}$) is isomorphic with $\mathcal{A}$. To this end we need to topologize conveniently the rings $\mathcal{A}(U)$, where U is a coordinate chart for the (m, n) dimensional $\mathcal{G}$-supermanifold $(M, \mathcal{A})$ in the sense of Definition 2. Let $\| \, \|$ denote the l^1 norm on B_L; we define on $\mathcal{A}(U)$ the family of seminorms

$$|f|_K^r = \max_{x \in U} \sum_{\substack{|\alpha| \leq r \\ \beta \in \Xi_n}} \left\| \delta \left(\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} f \right)(x) \right\|, \qquad r \in \mathbb{N} \tag{3.1}$$

where K is any compact subset of U, α is a multi-index with $|\alpha|$ its length, (x, y) are respectively even and odd coordinates on U, and

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial}{\partial y^{\alpha_1}} \cdots \frac{\partial}{\partial y^{\alpha_N}}$$

$$\frac{\partial}{\partial y^\beta} = \frac{\partial}{\partial y^{\beta(1)}} \cdots \frac{\partial}{\partial y^{\beta(s)}} \qquad \text{if} \quad \beta = \{\beta_1 \ldots \beta(s)\}.$$

In this way $\mathcal{A}(U)$ is a Fréchet algebra.

Now, let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be $\mathcal{G}$-supermanifolds.

DEFINITION 3. *A morphism of $\mathcal{G}$-supermanifold is a morphism $(f, \Phi) : (M, \mathcal{A}) \to (N, \mathcal{B})$ of ringed spaces of graded B_L-algebras preserving the G^∞ structure. That is, $f : M \to N$ is a G^∞ morphism and $\Phi : \mathcal{B} \to f_* \mathcal{A}$ is an even morphism of graded B_L-algebras such that there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & f_* \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{B}^\infty & \xrightarrow{f^*} & f_* \mathcal{A}^\infty \end{array} \tag{3.2}$$

(2) The notion of $\mathcal{G}$-maps will be formally defined hereunder.

where $f^\star$ stands for composition with f. Moreover, Φ is required to be continuous in the topology induced by the seminorms (3.1).

Thus $\mathcal{G}$-morphisms are not completely characterized by the G^∞ map $f : M \to N$ they define; indeed the sheaf morphism $\Phi : \mathcal{B} \to f_\star\mathcal{A}$ encodes further information, as the following example shows.

EXAMPLE 3.1. Considering the case $M = N = B_L \equiv B_L^{1,1}$, both with structure sheaf $\mathcal{G}$, we can define two different $\mathcal{G}$-morphism (f, ϕ) and (f, ψ), having the same underlying G^∞ map f. Let $f : B_L \to B_L$ be the GH^∞ map $f(x, y) = (x, 0)$, and let a be an even top-degree element in B_L (obviously, we assume that L is even). The condition $\phi(g \otimes \lambda) = (f^\star g) \otimes \lambda$ defines a $\mathcal{G}$-morphism $(f, \phi) : (B_L, \mathcal{G}) \to (B_L, \mathcal{G})$.

A second morphism $\psi : \mathcal{G} \to f_\star\mathcal{G}$ can be defined by

$$\psi(g \otimes \lambda) = \alpha \otimes \lambda + \hat{\beta} \otimes a\lambda,$$

having set $g(x, y) = \alpha(x) + y\beta(x)$ and $\hat{\beta}(x, y) = y\beta(x)$. Simple direct calculations show that $\delta \circ \psi = f^\star \circ \delta$, and that ψ is a continuous morphism of graded B_L-algebras. Thus, (f, ψ) is another $\mathcal{G}$-morphism, with the same underlying G^∞ (actually, GH^∞) map as (f, ϕ).

We denote by $Hom_{\mathcal{G}}(M, N)$ the sheaf of germs of $\mathcal{G}$-maps $(M, \mathcal{A}) \to (N, \mathcal{B})$. In particular we are interested in the case $(N, \mathcal{B}) \equiv (B_L, \mathcal{G})$. For every open subset $U \subset M$ we have morphisms of graded B_L-modules

$$\gamma : Hom_{\mathcal{G}}(U, B_L) \to \mathcal{A}(U)$$
$$(f, \Phi) \mapsto \Phi(j \otimes 1)$$

where j is the inclusion $f(U) \to B_L$, so that $\Phi(j \otimes 1) \in \mathcal{A}(U)$. Thus we get a morphism of sheaves of graded commutative B_L-algebras

$$Hom_{\mathcal{G}}(M, B_L) \to \mathcal{A}. \tag{3.3}$$

THEOREM. *The morphism (3.3) is an isomorphism.*

In other words, even though the «abstract» sheaf $\mathcal{A}$ is not a sheaf of functions, in that it is strictly larger than the sheaf $\mathcal{A}^\infty$ of G^∞ maps $M \to B_L$, according to the exact sequence [6]

$$0 \to \mathcal{N}^{L+1} \to \mathcal{A} \to \mathcal{A}^\infty \to 0 \tag{3.4}$$

nevertheless it may be identified with the sheaf of $\mathcal{G}$-morphisms from $(M, \mathcal{A})$ to $(B_L, \mathcal{G})$ (in the sequence (3.4) $\mathcal{N}^{L+1}$ denotes the $(L + 1) - th$ graded symmetric power of the nilpotent sheaf of $\mathcal{A}^\infty$).

Proof of the Theorem. Since we are dealing with sheaves, it is enough to prove the theorem for $(M,\mathcal{A}) = (B_L^{m,n},\mathcal{G})$, showing that for every open subset $U \subset B_L^{m,n}$ the map γ is an isomorphism of graded B_L-modules.

Given an element

$$h = \sum_i h_i \otimes \xi_i \in \mathcal{G}(U) \equiv \mathcal{GH}(U) \otimes_{B_{L'}} B_L,$$

$$h_i \in \mathcal{GH}(U), \qquad \xi_i \in B_L,$$

one defines a $\mathcal{G}$-morphism $(\tilde{h},\bar{h}) : (U,\mathcal{G}_U) \to (B_L,\mathcal{G}_{B_L})$ by taking $\tilde{h} : U \to B_L$ as the G^∞ map $\tilde{h} = \sum_i h_i \xi_i$ induced by h, and $\bar{h} : \mathcal{G}_{B_L} \to \mathcal{G}_U$ as the morphism described by

$$\bar{h}(\sum_k g_k \otimes \lambda_k) = \sum_{i,k} (-1)^{|h_i||\lambda_k|} (g_k \circ h_i) \otimes \lambda_k \xi_i .$$

This defines a map $\chi : \mathcal{G}(U) \to Hom_{\mathcal{G}}(U,B_L)$, given by $\chi(h) = (\tilde{h},\bar{h})$, such that $\gamma \circ \chi$ is the identity on $\mathcal{G}(U)$. One need also proving that $\chi \circ \gamma$ is the identity, which amounts to showing that a morphism $(f,\Phi) \in Hom_{\mathcal{G}}(U,B_L)$ is characterized by $\Phi(j \otimes 1)$. But if x, y are the coordinates in B_L considered as GH^∞ maps from B_L to B_L, one has $j \otimes 1 = x \otimes 1 + y \otimes 1$ as elements in $\mathcal{G}(B_L)$ and so $\Phi(j \otimes 1) = \Phi(x \otimes 1) + \Phi(y \otimes 1)$. Since Φ is even, $\Phi(x \otimes 1)$ (resp. $\Phi(y \otimes 1)$) is the even (resp. odd) part of $\Phi(j \otimes 1)$. This proves that if one knows the element $\Phi(j \otimes 1)$, then also the elements $\Phi(x \otimes 1)$ and $\Phi(y \otimes 1)$ are known and hence, since Φ is continuous by hypothesis, one can calculate $\Phi(f \otimes \xi)$ for every $f \in \mathcal{G}(B_L)$, $\xi \in B_L$ thus concluding the proof. □

As noticed at the end of Section 2, any GH^∞ supermanifold yields a $\mathcal{G}$-supermanifold; one can wonder whether there are $\mathcal{G}$-supermanifolds which are not obtained by extending GH^∞ supermanifolds. This is equivalent to the existence of $\mathcal{G}$-maps which are not GH^∞ maps. It is not difficult to construct such maps.

EXAMPLE 3.2. Let $f : B_L^{1,0} \to B_L^{1,0}$ be a G^∞ map which is not GH^∞, e.g. $f(x) = xa$ with $a \in B_L$ but $a \notin B_{L'}$. Define $\Phi : \mathcal{G} \to f_* \mathcal{G}$ by letting

$$\Phi(g) = \sum_{k=0}^{L} \sum_i \frac{1}{k!} \frac{\partial^k g_i}{\partial x^k}(0)\, x^k \otimes a^k \lambda_i \;\text{ if }\; g = \sum_i g_i \otimes \lambda_i .$$

In this case the commutativity of (3.2) amounts to

$$\delta(\Phi(g)) = f^*(\delta(g)) \qquad \text{i.e.} \qquad \sum_i f^*(g_i)\lambda_i = f^*\left(\sum_i g_i \lambda_i\right)$$

which holds since f is G^∞. Thus $(f,\Phi) : (B_L,\mathcal{G}) \to (B_L,\mathcal{G})$ is a $\mathcal{G}$-morphism, whilst evidently it is not GH^∞. □

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